MAHALAKSHMI
ENGINEERING COLLEGE
TIRUCHIRAPALLI – 621213

DEPARTMENT: CIVIL **SEMESTER: VII**

SUB.CODE/ NAME: CE2403/ BASICS OF DYNAMICS AND ASEISMIC DESIGN

UNIT – II

MULTIPLE DEGREE OF FREEDOM SYSTEM

Two degree of freedom system – Normal modes of vibration – Natural frequencies - Mode shapes - Introduction to MDOF systems – Decoupling of equations of motion – Concept of mode superposition (No derivations).

Two Marks Questions and Answers

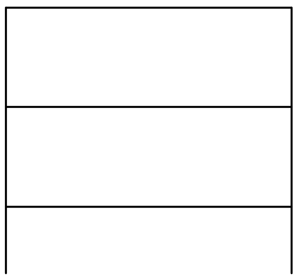
1. Define degrees of freedom.

The no. of independent displacements required to define the displaced positions of all the masses relative to their original position is called the no. of degrees of freedom for dynamic analysis.

2. Write a short note on matrix deflation technique.

Whenever the starting vector, the vector iteration method yields the same lowest Eigen value. To obtain the next lowest value, the one already found must be suppressed. This is possible by selecting vector that is orthogonal to the eigen values already found, or by modifying any arbitrarily selected initial vector form orthogonal to already evaluated vectors. The Eigen vectors X_{L2} computed by iteration as in the previous example X_1 would be orthogonal to the X_{L1}. the corresponding frequency will be higher than λ_{L1} but lower than all other Eigen values.

3. Write the examples of multi degrees of freedom system.



4. What is mean by flexibility matrix?

Corresponding to the stiffness (k), there is another structural property known as flexibility which is nothing but the reciprocal of stiffness. The flexibility matrix F is thus the inverse of the stiffness matrix, $[F] = [K]^{-1}$.

5. Write a short note on Jacobi's Method.

While all other enable us to calculate the lowest Eigen values one after another, Jacobi's method yields all the Eigen values simultaneously. By a series of transformations of the classical form of the matrix prescribed by Jacobi, all the non diagonal terms may be annihilated, the final diagonal matrix gives all the Eigen values along the diagonal.

6. What are the steps to be followed to the dynamic analysis of structure?

The dynamic analysis of any structure basically consists of the following steps.

1. Idealize the structure for the purpose of analysis, as an assemblage of discreet elements which are interconnected at the nodal points.
2. Evaluate the stiffness, inertia and damping property matrices of the elements chosen.
3. By supporting the element property matrices appropriately, formulate the corresponding matrices representing the stiffness, inertia and damping of the whole structure.

7. Write a short note on Inertia force – Mass matrix [M]

On the same analogy, the inertia forces can be represented in terms of mass influence co efficient, the matrix representation of which is given by $\{f_1\} = [M]\{Y\}$

M_{ij} a typical element of matrix M is defined as the force corresponding to co – ordinate i due as the force corresponding to coordinate i due to unit acceleration applied to the co ordinate j. $[M]\{Y\}+[C]\{Y\}+[K]\{Y\} = \{P(t)\}$

8. What are the effects of Damping?

The presence of damping in the system affects the natural frequencies only to a marginal extent. It is conventional therefore to ignore damping in the computations for natural frequencies and mode shapes

9. Write a short note on damping force – Damping force matrix.

If damping is assuming to be of the viscous type, the damping forces may likewise be represented by means of a general damping influence co efficient, C_{ij}. In matrix form this can be represented as

$$\{fD\}= [C] \{Y\}$$

10. What are the steps to be followed to the dynamic analysis of structure?

The dynamic analysis of any structure basically consists of the following steps.

1. Idealize the structure for the purpose of analysis, as an assemblage of discreet elements which are interconnected at the nodal points.
2. Evaluate the stiffness, inertia and damping property matrices of the elements chosen.
3. By supporting the element property matrices appropriately, formulate the corresponding matrices representing the stiffness, inertia and damping of the whole structure.

11. What are normal modes of vibration?

If in the principal mode of vibration, the amplitude of one of the masses is unity, it is known as normal modes of vibration.

12. Define Shear building.

Shear building is defined as a structure in which no rotation of a horizontal member at the floor level. Since all the horizontal members are restrained against rotation, the structure behaves like a cantilever beam which is deflected only by shear force.

13. What is mass matrix?

The matrix $\begin{bmatrix} m_1 & \\ 0 & \end{bmatrix}$ is called mass matrix and it can also be represented as [m]

14. What is stiffness matrix?

The matrix $\begin{bmatrix} k_1 + k_2 & - \\ -k_2 & \end{bmatrix}$ is called stiffness matrix and it also denoted by [k]

15. Write short notes on orthogonality principles.

The mode shapes or Eigen vectors are mutually orthogonal with respect to the mass and stiffness matrices. Orthogonality is the important property of the normal modes or Eigen vectors and it used to uncouple the modal mass and stiffness matrices.

$\{\phi\}_i^T[k]\{$ = 0, this condition is called orthogonality principles.

16. Explain Damped system.

The response to the damped MDOF system subjected to free vibration is governed by $[M]\{\ddot{u}\} + [c]\{\dot{u}\} + [k]\{u\}$

In which [c] is damping matrix and { } is velocity vector.

Generally small amount of damping is always present in real structure and it does not have much influence on the determination of natural frequencies and mode shapes of the system. The naturally frequencies and mode shapes for the damped system are calculated by using the same procedure adopted for undamped system

17. What is meant by first and second mode of vibration?

The lowest frequency of the vibration is called fundamental frequency and the corresponding displacement shape of the vibration is called **first mode or fundamental mode of vibration**. The displacement shape corresponding to second higher natural frequency is called **second mode of vibration.**

18. Write the equation of motion for an undamped two degree of freedom system.

$$[m]\{\ddot{u}\} + [k]\{u\}$$

This is called equation of motion for an undamped two degree of freedom system subjected to free vibration.

19. What is meant by two degree of freedom and multi degree of freedom system?

The system which requires two independent coordinates to describe the motion is completely is called two degree of freedom system. In general, a system requires n number of independent coordinates to describe it motion is called multi degree of freedom system

20. Write the characteristic equation for free vibration of undamped system.

$$|[k] - \omega^2[m]|$$

This equation is called as characteristic equation or frequency equation.

16 MARKS

SOLVED EXAMPLES

EXAMPLE 7.1 Determine the natural frequencies and mode shape of the given system.

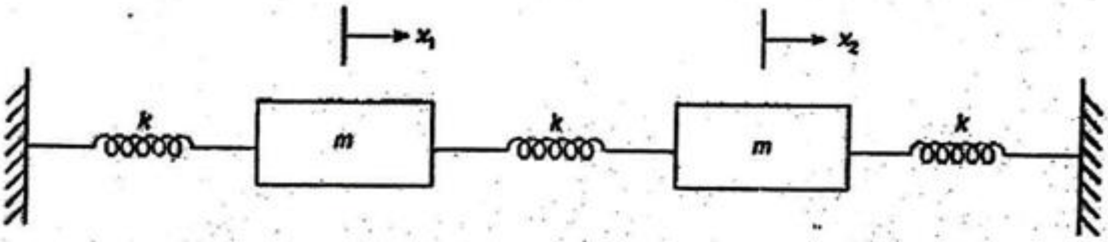


Figure 7.7

Solution:

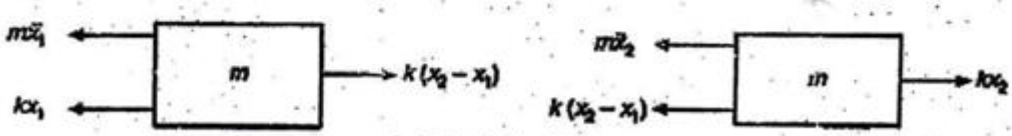


Figure 7.8 FBD.

The governing differential equation of motion can be written as

$$m\ddot{x}_1 + kx_1 - kx_2 + kx_1 = 0 \quad \text{or} \quad m\ddot{x}_1 + 2kx_1 - kx_2 = 0$$

$$m\ddot{x}_2 + kx_2 - kx_1 + kx_2 = 0 \quad \text{or} \quad m\ddot{x}_2 - kx_1 + 2kx_2 = 0$$

Thus, $\omega_1^2 = k/m$ (Take lesser value for fundamental mode)

$$\omega_2^2 = \frac{3k}{m}$$

Hence the two natural frequencies are,

$$\omega_1 = \sqrt{\frac{k}{m}}$$

$$\omega_2 = \sqrt{\frac{3k}{m}}$$

First mode

To find the first mode shape, substitute the value of ω_1 in Eq. (2)

$$\left[\begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix} - \omega_n^2 \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}\right] \begin{Bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{Bmatrix} = 0 \qquad (2)$$

$$\Rightarrow \quad \left|k\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - m\omega_n^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right| \begin{Bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{Bmatrix} = 0$$

Substituting the value of $\omega_n^2 = \frac{k}{m}$, we get

$$\left[\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \frac{m}{k}\frac{k}{m}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]\begin{Bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{Bmatrix} = 0$$

$$\Rightarrow \quad \left[\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]\begin{Bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{Bmatrix} = 0 \qquad (3)$$

$$\Rightarrow \quad \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{Bmatrix} = 0$$

$$x_1^{(1)} - x_2^{(1)} = 0 \qquad (4)$$

$$-x_1^{(1)} + x_2^{(1)} = 0 \qquad (5)$$

$$\frac{x_1^{(1)}}{x_1^{(1)}} = 1$$

$$\frac{x_2^{(1)}}{x_1^{(1)}} = 1$$

Thus, the mode shape corresponding to the fundamental frequency is giveu as

$$= \begin{Bmatrix} \dfrac{x_1^{(1)}}{x_1^{(1)}} \\ \dfrac{x_2^{(1)}}{x_1^{(1)}} \end{Bmatrix}$$

$$\{\phi_1\} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

Figure 7.9 First mode shape.

Second mode

Substitute the value of ω_2^2 in Eq. (2) to obtain the second mode shape of vibration.

$$\Rightarrow \quad \left[\begin{matrix}2 & -1\\ -1 & 2\end{matrix}\right]-3\frac{k}{m}.\omega_n^2\left[\begin{matrix}1 & 0\\ 0 & 1\end{matrix}\right]\left\{\begin{matrix}x_1^{(2)}\\ x_2^{(2)}\end{matrix}\right\}=0$$

$$\Rightarrow \quad \left[\begin{matrix}2 & -1\\ -1 & 2\end{matrix}\right]-\left[\begin{matrix}3 & 0\\ 0 & 3\end{matrix}\right]\left\{\begin{matrix}x_1^{(2)}\\ x_2^{(2)}\end{matrix}\right\}=0$$

$$\Rightarrow \quad \left[\begin{matrix}-1 & -1\\ -1 & 1\end{matrix}\right]\left\{\begin{matrix}x_1^{(2)}\\ x_2^{(2)}\end{matrix}\right\}=0$$

$$\Rightarrow \quad -x_1^{(2)}-x_2^{(2)}=0$$

$$\Rightarrow \quad \frac{x_2^{(2)}}{x_1^{(2)}}=-1$$

$$\Rightarrow \quad \frac{x_1^{(2)}}{x_1^{(2)}}=1$$

Thus the mode shape corresponding to $\omega_2 = \left\{\begin{matrix}\frac{x_1^{(2)}}{x_1^{(2)}}\\ \frac{x_2^{(2)}}{x_1^{(2)}}\end{matrix}\right\}$

$$\{\phi_2\}=\left\{\begin{matrix}1\\ -1\end{matrix}\right\}$$

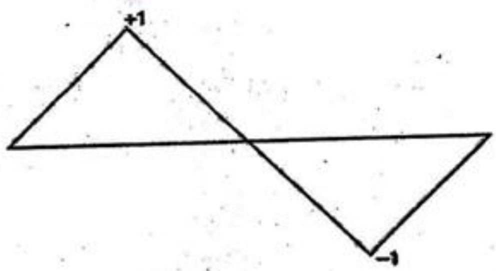


Figure 7.10 Second mode shape.

EXAMPLE 7.2 A cantilever bar is to be modelled by a massless uniform bar to which are attached with two lumped masses representing the mass of original system as $K = \frac{2AE}{L}$ and $m = \rho AL$. Determine the natural frequencies and the normal modes of this model.

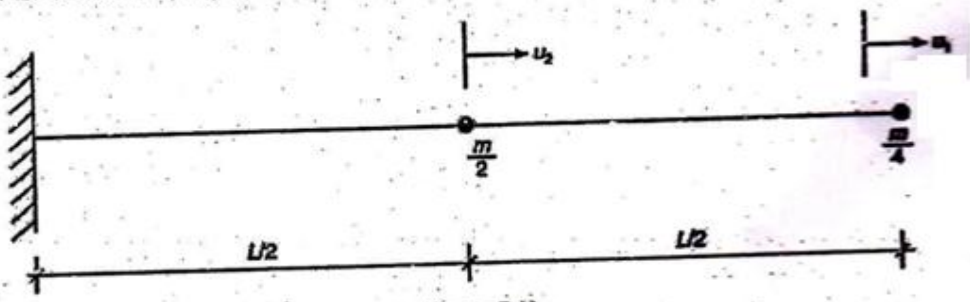


Figure 7.11

Solution:

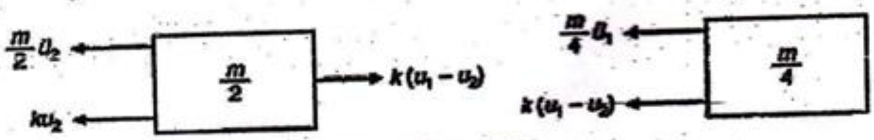


Figure 7.12 FBD.

The governing differential equations of motion can be written as

$$\frac{m}{2}\ddot{u}_2 + k(u_2 - 0) - k(u_1 - u_2) = 0$$

$\Rightarrow$
$$\frac{m}{4}\ddot{u}_1 + k(u_1 - u_2) = 0$$

These two equations can be rearranged as,

$$\frac{m}{2}\ddot{u}_2 + 2ku_2 - ku_1 = 0 \qquad (1)$$

$\Rightarrow$
$$\frac{m}{4}\ddot{u}_1 + ku_1 - ku_2 = 0 \qquad (2)$$

Writing the above two equations into matrix form after rearranging

$$\begin{bmatrix} m/4 & 0 \\ 0 & m/2 \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} k & -k \\ -k & 2k \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \{0\}$$

$\because \; m = \rho AL; \; K = \frac{2AE}{L}$

$\Rightarrow$
$$\begin{bmatrix} \frac{\rho AL}{4} & 0 \\ 0 & \frac{\rho AL}{2} \end{bmatrix}\begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix} + \begin{bmatrix} 2AE/L & -2AE/L \\ -2AE/L & 4AE/L \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \{0\}$$

$$\Rightarrow \quad \frac{\rho AL}{4}\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\begin{Bmatrix}\ddot{u}_1\\\ddot{u}_2\end{Bmatrix}+\frac{2AE}{L}\begin{bmatrix}1 & -1\\-1 & 2\end{bmatrix}\begin{Bmatrix}u_1\\u_2\end{Bmatrix}=0$$

Dividing by $\frac{\rho AL}{4}$

$$\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}+\frac{8E}{\rho L^2}\begin{bmatrix}1 & -1\\-1 & 2\end{bmatrix}\begin{Bmatrix}u_1\\u_2\end{Bmatrix}=0$$

Characteristic equation is

$$\left|[k]-\omega_n^2[m]\right|=0$$

$$\Rightarrow \quad \left|\frac{8E}{\rho L^2}\begin{bmatrix}+1 & -1\\-1 & 2\end{bmatrix}-\omega_n^2\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\right|=0$$

Dividing by $8E/\rho L^2$

$$\left|\begin{bmatrix}1 & -1\\-1 & 2\end{bmatrix}-\omega_n^2\frac{\rho L^2}{8E}\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\right|=0$$

Let

$$\lambda=\frac{\omega_n^2\rho L^2}{8E}$$

$$\left|\begin{bmatrix}1 & -1\\-1 & 2\end{bmatrix}-\lambda\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}\right|=0$$

$$\Rightarrow \quad \begin{vmatrix}1-\lambda & -1\\-1 & 2-2\lambda\end{vmatrix}=0$$

$$\Rightarrow \quad (1-\lambda)(2-2\lambda)-1=0$$

$$\Rightarrow \quad 2\lambda^2-4\lambda+1=0$$

$$\Rightarrow \quad \lambda_{1,2}=0.293, \quad 1.707$$

$$\Rightarrow \quad \lambda_1=0.293=\frac{\omega_1^2\rho L^2}{8E}$$

Therefore,

$$\omega_1^2=0.293\frac{(8E)}{\rho L^2}$$

$$\omega_1^2=2.344\frac{E}{\rho L^2}$$

or

$$\omega_1=\frac{1.53}{L}\sqrt{\frac{E}{\rho}}$$

Similarly,

$$\lambda_2 = 1.707 = \frac{\omega_2^2 \rho L^2}{8E}$$

$$\omega_2^2 = 1.707\left(\frac{8E}{\rho L^2}\right)$$

$\Rightarrow$

$$\omega_2^2 = 13.656\frac{E}{\rho L^2}$$

$\Rightarrow$

$$\omega_2 = \frac{3.695}{L}\sqrt{\frac{E}{\rho}}$$

The natural frequencies are,

$$\omega_1 = \frac{1.53}{L}\sqrt{\frac{E}{\rho}}$$

$$\omega_2 = \frac{3.695}{L}\sqrt{\frac{E}{\rho}}$$

First mode

Substituting the value of ω_1 to get the first mode of vibration.

$$\begin{vmatrix} 1-\lambda_1 & -1 \\ -1 & 2-2\lambda_1 \end{vmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = 0$$

$$\lambda_1 = 0.293$$

$\Rightarrow$

$$\begin{vmatrix} 1-0.293 & -1 \\ -1 & 2-2(0.293) \end{vmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = 0$$

$\Rightarrow$

$$\begin{vmatrix} 0.707 & -1 \\ -1 & 2.586 \end{vmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = 0$$

$\Rightarrow$

$$0.707u_1 - u_2 = 0$$

$$\frac{u_2}{u_1} = 0.707$$

$$\frac{u_1^{(1)}}{u_1^{(1)}} = 1$$

Thus, the mode shape corresponding to ω_1 is

$$\phi_1 = \begin{Bmatrix} 1 \\ 0.707 \end{Bmatrix}$$

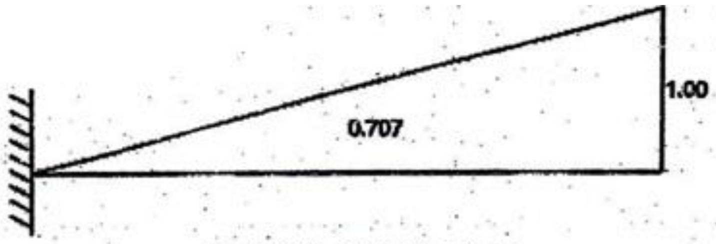


Figure 7.13 First mode shape.

Second mode

Substitute the value of ω_2 to get the second mode shape.

$$\begin{vmatrix} 1-\lambda_2 & -1 \\ -1 & 2-2\lambda_2 \end{vmatrix} \begin{Bmatrix} u_1^{(2)} \\ u_2^{(2)} \end{Bmatrix} = 0$$

$$\lambda_2 = 1.707$$

$$\Rightarrow \qquad \begin{vmatrix} 1-1.707 & -1 \\ -1 & 2-2(1.707) \end{vmatrix} \begin{Bmatrix} u_1^{(2)} \\ u_2^{(2)} \end{Bmatrix} = 0$$

$$\Rightarrow \qquad \begin{vmatrix} -0.707 & -1 \\ -1 & -1.414 \end{vmatrix} \begin{Bmatrix} u_1^{(2)} \\ u_2^{(2)} \end{Bmatrix} = 0$$

$$\Rightarrow \qquad -0.707u_1 - u_2 = 0$$

$$\Rightarrow \qquad \begin{Bmatrix} u_2^{(2)} \\ u_1^{(2)} \end{Bmatrix} = -0.707$$

$$\Rightarrow \qquad \begin{Bmatrix} u_1^{(2)} \\ u_1^{(2)} \end{Bmatrix} = 1$$

The second mode shape is given by

$$\phi_2 = \begin{Bmatrix} 1 \\ -0.707 \end{Bmatrix}$$

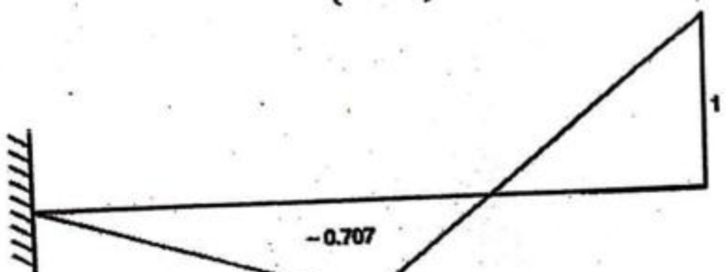


Figure 7.14 Second mode shape.

EXAMPLE 7.3 Determine the natural frequencies and mode shape for the structure as shown in Figure 7.15.

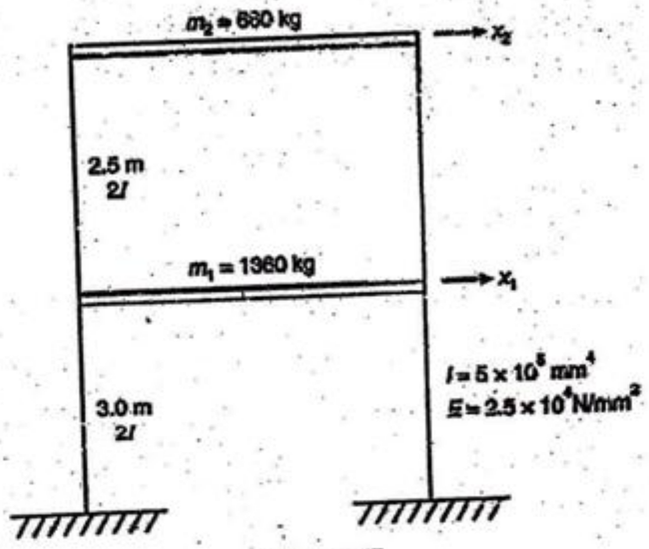


Figure 7.15

Solution: Equivalent system for the above structure is

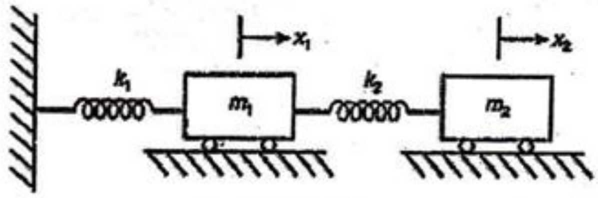


Figure 7.16

Stiffness $k = \dfrac{12EI}{L^3}$

$$\Rightarrow \qquad k_1 = \frac{12\times25000\times2\times500\times10^3}{(3000)^3}$$

$$= 11.11 \text{ N/mm}$$

$$= 11.11\times10^3 \text{ N/m}$$

$$k_2 = \frac{12\times25000\times2\times500\times10^3}{(2500)^3} = 19.2 \text{ N/mm}$$

$$= 19.2\times10^{-3} \text{ N/m}$$

The governing equation of motion can be written as,

$$m_1\ddot{x}_1 + k_1x_1 - k_2(x_2 - x_1) = 0$$

$$m_2\ddot{x}_2 + k_2x_2 - k_2x_1 = 0$$

Rearranging the above equations, we get

$$m_1\ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0$$

$$m_2\ddot{x}_2 + k_2x_2 - k_2x_1 = 0$$

Writing the above equations into matrix form

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0$$

The characteristic equation is

$$\left|[k] - \omega_n^2[m]\right| = 0$$

$\Rightarrow$

$$\begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} - \omega_n^2\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} = 0$$

$\Rightarrow$

$$\begin{vmatrix} k_1 + k_2 - m_1\omega_n^2 & -k_2 \\ -k_2 & k_2 - m_2\omega_n^2 \end{vmatrix} = 0$$

By expanding the above determinant, we get

$$m_1m_2\omega_n^4 - \{k_2m_1 + (k_1 + k_2)m_2\}\omega_n^2 + k_1k_2 = 0$$

Substituting the values of m_1, m_2, k_1 and k_2, we get

$$897600\omega_n^4 - 46.116\times10^6\omega_n^2 + 213.31\times10^6 = 0$$

$$\omega_n^4 - 51.38\omega_n^2 + 237.65 = 0$$

$\Rightarrow$

Let $s = \omega_n^2$

$$s^2 - 51.38s + 237.65 = 0$$

$$s_{1,2} = 46.24, 5.139$$

$$\omega_1^2 = s_1 = 5.139$$

$$\omega_1 = 2.27 \text{ rad/s}$$

$$\omega_2^2 = s_2 = 46.24$$

$$\omega_2 = 6.8 \text{ rad/s}$$

The two natural frequencies are,

$$\omega_1 = 2.27 \text{ rad/s}$$

$$\omega_2 = 6.8 \text{ rad/s}$$

First mode shape

Substitute the value of $\omega_1 = 0.026$ rad/s to get the first mode shape

$$\begin{vmatrix} k_1 + k_2 - m_1\omega_1^2 & -k_2 \\ -k_2 & k_2 - m_2\omega_1^2 \end{vmatrix} \begin{Bmatrix} x_1^{(1)} \\ x_2^{(2)} \end{Bmatrix} = 0$$

Substituting the values of k_1, k_2, m_1, m_2, and ω_1^2, we get

$$\begin{vmatrix} 23321 & -19.2\times10^3 \\ -19.2\times10^3 & 15808 \end{vmatrix} \begin{Bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{Bmatrix} = 0$$

$$368.6\times10^6\, x_1^{(1)} - 368.6\times10^6\, x_2^{(1)} = 0$$

$$\Rightarrow \qquad \frac{x_2^{(1)}}{x_1^{(1)}} = 1.00$$

Thus, the first mode shape is

$$\{\phi_1\} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

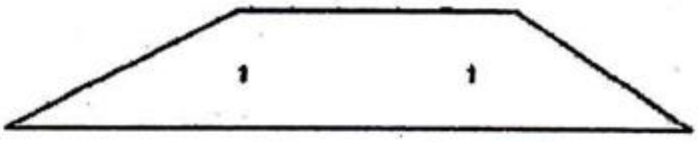


Figura 7.17 First mode shape.

Second mode shape

Substitute the value of ω_2 to obtain the second mode shape

$$\begin{vmatrix} k_1 + k_2 - m_1\omega_2^2 & -k_2 \\ -k_2 & k_2 - m_2\omega_2^2 \end{vmatrix} \begin{Bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{Bmatrix} = 0$$

Substituting the values of k_1, k_2, m_1, m_2, and ω_2^2, we get

$$\begin{vmatrix} -32.57\times10^3 & -19.2\times10^3 \\ -19.2\times10^3 & -11.31\times10^3 \end{vmatrix} \begin{Bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{Bmatrix} = 0$$

$$\Rightarrow \qquad -32.57\times 10^3\, x_1^{(2)} - 19.2\times 10^3 x_2^{(2)} = 0$$

$$\Rightarrow \qquad \frac{x_2^{(2)}}{x_1^{(2)}} = -1.697$$

The second mode shape is

$$\{\phi_2\} = \begin{Bmatrix} 1 \\ -1.697 \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1.70 \end{Bmatrix}$$

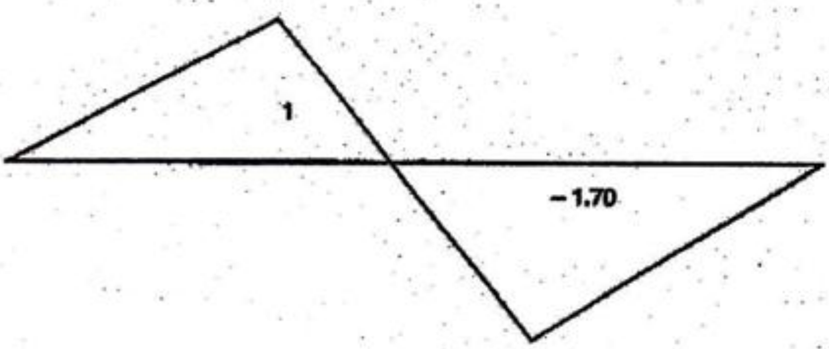


Figure 7.18 Second mode shape.

EXAMPLE 7.4 Determine the natural frequencies and mode of vibration of the given system.

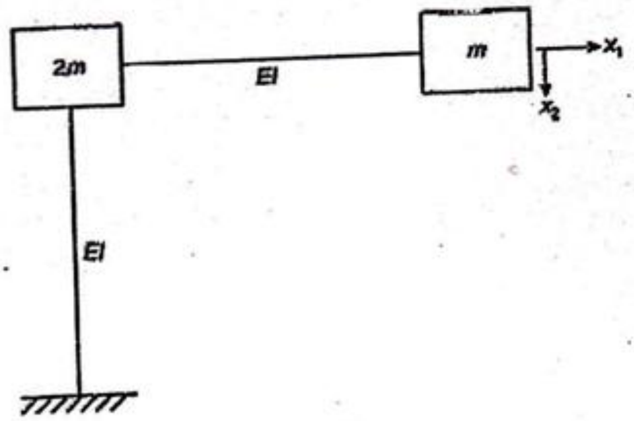


Figure 7.19

Both the mass m and $2m$ are contributes for horizontal displacement. That is, $\ddot{x}_1 = 1$, the mass corresponding to this is $2m + m = 3m$. But only the mass m will be contributed for vertical displacement. That is,

$\ddot{x}_2 = 1$ the mass corresponding to this is only m.

Thus, the mass matrix will be $\begin{bmatrix} 2m+m & 0 \\ 0 & m \end{bmatrix} = \begin{bmatrix} 3m & 0 \\ 0 & m \end{bmatrix}$

By using matrix method of structural analysis, the flexibility matrix is $[f] = \frac{L^3}{6EI}\begin{bmatrix} 2 & 3 \\ 3 & 8 \end{bmatrix}$

$\therefore$ Stiffness matrix $[k] = \frac{6EI}{7L^3}\begin{bmatrix} 8 & -3 \\ -3 & 2 \end{bmatrix}$

Thus equation of motion can be written as

$$\begin{bmatrix} 3m & 0 \\ 0 & m \end{bmatrix}\begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \frac{6EI}{7L^3}\begin{bmatrix} 8 & -3 \\ -3 & 2 \end{bmatrix}\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = 0$$

Characteristic equation is $\left|[k] - \omega_n^2[m]\right| = 0$

$$\Rightarrow \quad \left|\frac{6EI}{7L^3}\begin{bmatrix} 8 & -3 \\ -3 & 2 \end{bmatrix} - \omega_n^2\begin{bmatrix} 3m & 0 \\ 0 & m \end{bmatrix}\right| = 0$$

Dividing by $\frac{6EI}{7L^3}$

$$\left|\begin{bmatrix} 8 & -3 \\ -3 & 2 \end{bmatrix} - \frac{7L^3}{6EI}\omega_n^2 m\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0$$

Let $\lambda = \frac{7L^3\omega_n^2 m}{6EI}$. Then the above equation becomes.

$$\left|\begin{bmatrix} 8 & -3 \\ -3 & 2 \end{bmatrix} - \lambda\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\right| = 0$$

$$\Rightarrow \quad \begin{vmatrix} 8-3\lambda & -3 \\ -3 & 2-\pi \end{vmatrix} = 0$$

Expanding the above determinant, we get

$$(8-3\lambda)(2-\lambda) - 9 = 0$$

$$\Rightarrow \quad 3\lambda^2 - 14\lambda + 7 = 0$$

$$\Rightarrow \quad \lambda_{1,2} = 0.5695,\ 4.097$$

$$\Rightarrow \quad \lambda_1 = 0.5695\frac{7L^3\omega_1^2 m}{6EI}; \quad \lambda_2 = 4.097\frac{7L^3\omega_2^2 m}{6EI}$$

$$\omega_1 = 0.6987\sqrt{\frac{EI}{mL^3}}; \quad \omega_2 = 1.874\sqrt{\frac{EI}{mL^3}}$$

Thus, natural frequencies are

$$\omega_1 = 0.6987\sqrt{\frac{EI}{mL^3}}$$

$$\omega_2 = 1.874\sqrt{\frac{EI}{mL^3}}$$

Mode shape

First mode

Substitute the value of ω_1 to obtain the first mode of vibration.

$$\omega_1 = 0.6987\sqrt{\frac{EI}{mL^3}}$$

$$\lambda_1 = 0.5695$$

$$\Rightarrow \begin{bmatrix} 8 & -3 \\ -3 & 2 \end{bmatrix} - 0.5695\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{Bmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} 6.2915 & -3 \\ -3 & 1.4305 \end{vmatrix}\begin{Bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{Bmatrix} = 0$$

$$\Rightarrow 6.2915\ x_1^{(1)} - 3x_2^{(1)} = 0$$

$$\frac{x_2^{(1)}}{x_1^{(1)}} = 2.0971$$

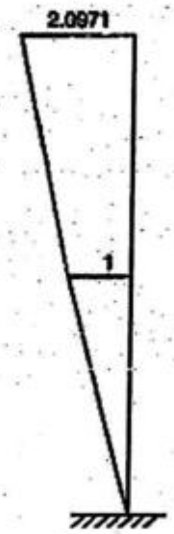


Figure.7.20 First mode shape.

Thus the mode shape corresponding to ω_1 is

$$\phi_1 = \begin{Bmatrix} 1 \\ 2.0971 \end{Bmatrix}$$

Second mode

Substitute the value of ω_1 to get second mode of vibration

$$\omega_2 = 1.874\sqrt{\frac{EI}{mL^3}}$$

$$\lambda_2 = 4.0972$$

$$\begin{bmatrix} 8 & -3 \\ -3 & 2 \end{bmatrix} - 4.0972\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{Bmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} -4.2916 & -3 \\ -3 & -2.0972 \end{vmatrix}\begin{Bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{Bmatrix} = 0$$

$$-4.2916x_1^{(2)} - 3x_2^{(2)} = 0$$

$$\frac{x_2^{(2)}}{x_1^{(2)}} = -1.431$$

Figure 7.21 Second mode shape.

Thus, the second mode shape is

$$\{\phi_2\} = \begin{Bmatrix} 1 \\ -1.431 \end{Bmatrix}$$